Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY

Mon. Not. R. Astron. Soc. **406,** 548–550 (2010) doi:10.1111/j.1365-2966.2010.16686.x

An analytical approach for the determination of the luminosity distance in a flat universe with dark energy

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Accepted 2010 March 14. Received 2010 March 14; in original form 2010 March 1

ABSTRACT

Recent cosmological observations indicate that the present universe is flat and dark energy dominated. In such a universe, the calculation of the luminosity distance, d_L, involves repeated numerical calculations. In this paper, it is shown that a quite efficient approximate analytical expression, having very small uncertainties, can be obtained for d_L. The analytical calculation is shown to be exceedingly efficient, as compared to the traditional numerical methods, and is potentially useful for Monte Carlo simulations involving luminosity distances.

Key words: distance scale.

1 INTRODUCTION

The most recent cosmological observations indicate that the present universe is flat and vacuum dominated (Komatsu et al. 2009). In such a vacuum-dominated space–time, the distance analysis requires computer-intensive numerical calculations. Even though today's computers are very fast, efficient analytical calculation of distance scales would be very useful for various types of Monte Carlo simulations.

The most fundamental distance scale in the universe is the luminosity distance, defined by $d_L = \sqrt{L/(4\pi f)}$, where f is the observed flux of an astronomical object and L is its luminosity. Current astronomical observations indicate that the present density parameter of the universe satisfies $\Omega_\Lambda + \Omega_M = 1$ with $\Omega_\Lambda \sim 0.7$. Here, Ω_Λ is the contribution from the vacuum and Ω_M is the contribution from all other fields. The distance calculations in such a vacuum-dominated universe involve repeated numerical calculations and elliptic functions (Eisenstein 1997).

In order to simplify the numerical calculations, Pen (1999, hereafter Pen99) has developed quite an efficient analytical recipe. In this paper, we show another analytical method, similar in many respects to that of Pen99, that can be used to calculate the distances in a vacuum-dominated flat universe.

Our analytical calculation is shown to run faster than that of Pen99 and has smaller error variations with respect to redshift (z) and Ω_Λ.

Our recipe for calculating the luminosity distance is as follows (H_0 is the present Hubble constant and c is the speed of light):

$$d_L = \frac{c}{3H_0} \frac{1+z}{\Omega_\Lambda^{1/6}(1-\Omega_\Lambda)^{1/3}} [\Psi(x(0,\Omega_\Lambda)) - \Psi(x(z,\Omega_\Lambda))], \quad (1)$$

$$\Psi(x) = 3\,x^{1/3} 2^{2/3} \left[1 - \frac{x^2}{252} + \frac{x^4}{21\,060}\right], \quad (2)$$

$$x = x(z,\Omega_\Lambda) = \ln(\alpha + \sqrt{\alpha^2 - 1}), \quad (3)$$

$$\alpha = \alpha(z,\Omega_\Lambda) = 1 + 2\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\frac{1}{(1+z)^3}. \quad (4)$$

2 APPROXIMATION

We first begin by analysing how the scalefactor, $a(t)$, varies as a function of time t in a flat universe in which $\Omega_\Lambda \neq 0$. In this case, $a(t)$ is given by (Weinberg 2008)

$$\dot{a}^2 = H_0^2 \Omega_\Lambda a^2 + H_0^2 \Omega_m \frac{a_0^3}{a}, \quad (5)$$

where a_0 is the present value of the scalefactor. The above equation is then immediately integrated into

$$\left(\frac{a}{a_0}\right)^3 = \frac{1}{2}\frac{\Omega_m}{\Omega_\Lambda}\left[\cosh(3H_0 t\sqrt{\Omega_\Lambda}) - 1\right]. \quad (6)$$

The scalefactor is directly related to z as

$$\frac{a}{a_0} = \frac{1}{1+z}. \quad (7)$$

Let us define $x = 3H_0 t\sqrt{\Omega_\Lambda}$ and indicate its present value by $x_0 = x(0,\Omega_\Lambda)$. Then, equations (6) and (7) give

$$x = x(z,\Omega_\Lambda) = \cosh^{-1}\left[1 + 2\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\frac{1}{(1+z)^3}\right]. \quad (8)$$

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If we define α as

$$\alpha = \alpha(z, \Omega_\Lambda) = 1 + 2\,\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\,\frac{1}{(1+z)^3} \tag{9}$$

and since $\alpha > 1$, we can write x as

$$x = x(z, \Omega_\Lambda) = \ln(\alpha + \sqrt{\alpha^2 - 1}). \tag{10}$$

We note that x is a monotonically decreasing function beyond $x(0, 0.7) = 2.42$.

We choose the standard Robertson–Walker metric (Weinberg 2008) as the metric of the background space–time. With usual notation, this is

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{11}$$

In the above space–time, we can use equation (5) to obtain r. A straightforward integration for a flat universe ($k = 0$) yields

$$r = \frac{c}{a_0 H_0}\,\frac{1}{3\Omega_\Lambda^{1/6}\Omega_\mathrm{M}^{1/3}}\int_x^{x_0}\frac{\mathrm{d}x'}{\left[\sinh\frac{x'}{2}\right]^{2/3}}. \tag{12}$$

This integral can be evaluated in terms of hypergeometric functions and related elliptic integrals. But here we take a simple, alternative approach by defining a new function

$$\Psi(x) = \lim_{\delta\to 0}\int_\delta^x \frac{\mathrm{d}x'}{\left[\sinh\frac{x'}{2}\right]^{2/3}}. \tag{13}$$

In the standard model, the luminosity distance is defined as $d_\mathrm{L} = a_0 r(1+z)$. Now we can use equation (13) to write the luminosity distance as

$$d_\mathrm{L} = \frac{c}{3H_0}\,\frac{1+z}{\Omega_\Lambda^{1/6}\Omega_\mathrm{M}^{1/3}}[\Psi(x_0) - \Psi(x)]. \tag{14}$$

Expanding Ψ in a series expansion to the fourth order, we find that

$$\Psi(x) = 3\,x^{1/3}2^{2/3}\left[1 - \frac{x^2}{252} + \frac{x^4}{21\,060}\right] + \Psi(0), \tag{15}$$

where $\Psi(0) = -2.210$. Now, equation (14) reduces to the required expression for the luminosity distance as

$$d_\mathrm{L} = \frac{c}{3H_0}\,\frac{1+z}{\Omega_\Lambda^{1/6}(1-\Omega_\Lambda)^{1/3}}[\Psi(x_0) - \Psi(x)]. \tag{16}$$

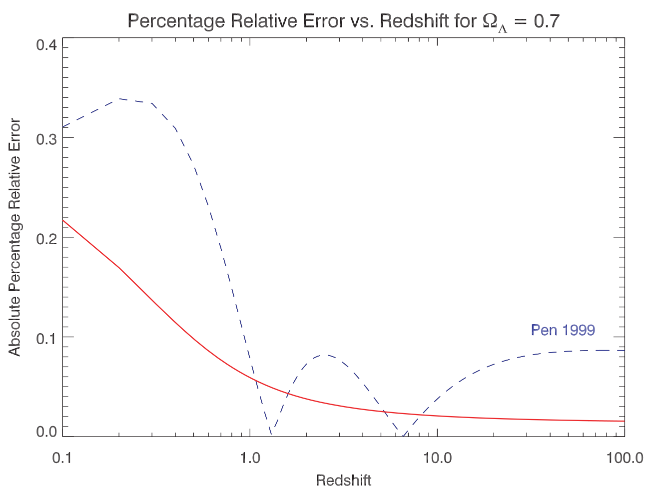

Figure 1. The absolute relative percentage error (ΔE) as a function of the redshift for $\Omega_\Lambda = 0.7$.

3 ANALYSIS AND CONCLUSION

In order to compare the method of Pen99 to ours, let us define the absolute relative percentage error as follows:

$$\Delta E = \frac{\left|d_\mathrm{L}^\mathrm{approx} - d_\mathrm{L}^\mathrm{num}\right|}{d_\mathrm{L}^\mathrm{num}} \times 100 \text{ per cent.} \tag{17}$$

Here $d_\mathrm{L}^\mathrm{approx}$ and $d_\mathrm{L}^\mathrm{num}$ are luminosity distance values calculated from approximate analytical methods and the numerical method, respectively.

A comparison of ΔE for both analytical methods for $\Omega_\Lambda = 0.7$ is shown in Fig. 1. Our method has a better absolute relative percentage error value for $z < 1.0$, $1.6 < z < 5.5$ and $z > 8.0$ compared to that of Pen99. We note that the error in our method decreases steadily with redshift approaching <0.014 per cent at $z = 1100$. In comparison, for high redshifts, the Pen99 error always stays ∼0.09 per cent and does not decrease appreciably.

A contour plot of ΔE based on the method of Pen99 with various z and Ω_Λ is shown in Fig. 2 . A relatively complicated distribution of variations in ΔE can be seen for the parameter space characterized by z and Ω_Λ. However, a contour plot of ΔE for our method, which is shown in Fig. 3 , shows a smooth behaviour over the same parameter space.

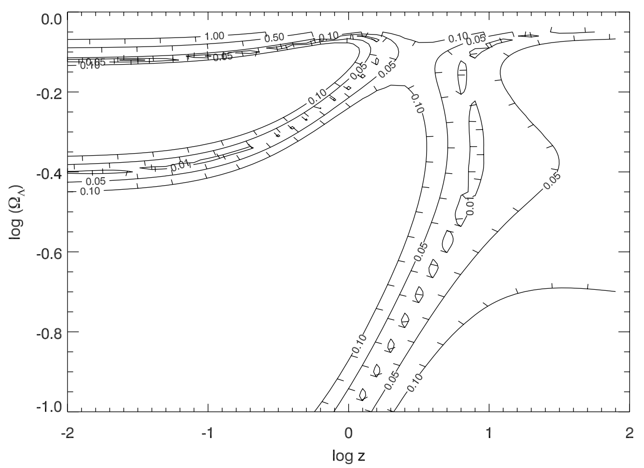

Figure 2. Contour plot of the absolute relative percentage error (ΔE) for the method of Pen99 with various z and Ω_Λ.

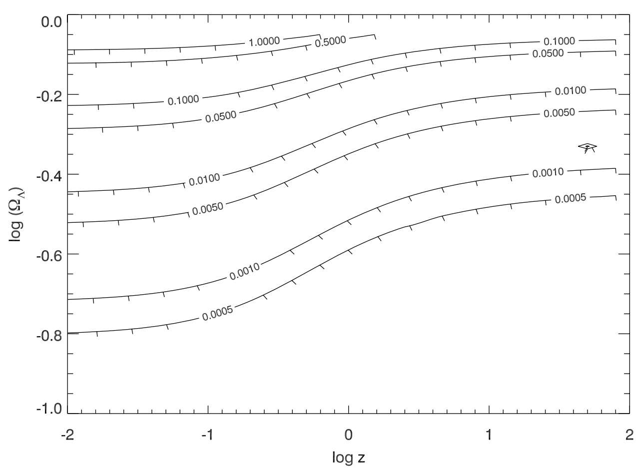

Figure 3. Contour plot of the absolute relative percentage error (ΔE) for our method with various z and Ω_Λ.

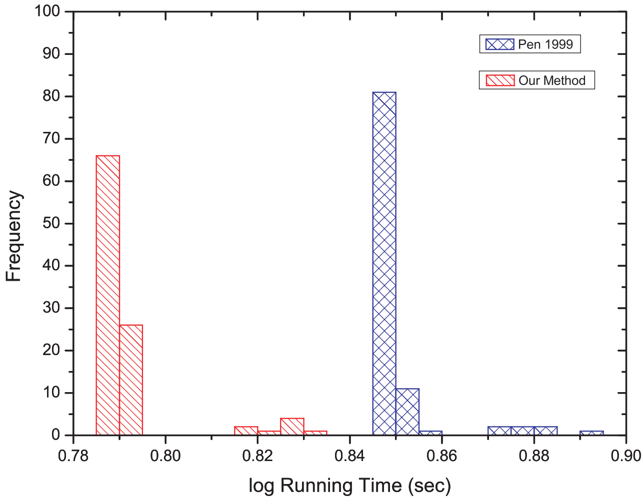

Figure 4. Histogram of running times of both Pen99 and our methods.

In order to investigate the running time of the two analytical methods, we performed the following test. With $z = 1$ and $\Omega_\Lambda = 0.7$, we calculated the running time for one million calculations on a typical personal computer (Intel Core 2 Processor, 2127 MHz, 1 GB RAM, IDL[1] Version 6.2 running on Windows XP Service Pack 3). Then we repeated the above process 100 times for the both methods. The histogram of both running time results is shown in Fig. 4. Our method is significantly faster than that of Pen99. In addition, we performed the same test on the numerical method and found that our method is more than an order of magnitude faster. However, we note that the above test is hardware and compiler dependent and results may vary depending on the hardware and the compiler used.

With less than 0.1 per cent error, our analytical method becomes quite desirable as the most interesting astronomical phenomena happen at $z > 1$ ($\Omega_\Lambda \sim 0.7$). Furthermore, the analytical computation is more elegant and faster compared to traditional numerical computations invoked in connection with calculations of distances in a vacuum-dominated flat universe.

Once we know the luminosity distance, it becomes a simple matter to evaluate the other distances such as the angular diameter distance or the proper distance.

[1] Interactive Data Language, http://www.ittvis.com/ProductServices/IDL.aspx.

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